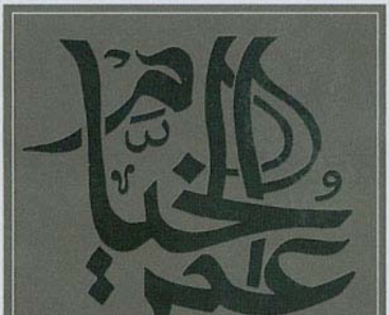
The Great Books of Islamic Civilization

An Essay by the Uniquely Wise 'Abel Fath Omar Bin Al-Khayyam on Algebra and Equations

Omar Al-Khayyam

Algebra wa Al-Muqabala

Translated by Professor Roshdi Khalil
Reviewed by Professor Waleed Deeb



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The Center for Muslim Contribution to Civilization

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In the Name of God, the Beneficent, the Merciful

Foreword

THE interrelationship and interaction of human cultures and civilizations has made the contributions of each the common heritage of men in all ages and all places. Early Muslim scholars were able to communicate with their Western counterparts through contacts made during the Crusades; at Muslim universities and centres of learning in Muslim Spain (al-Andalus, or Andalusia) and Sicily to which many European students went for education; and at the universities and centres of learning in Europe itself (such as Salerno, Padua, Montpellier, Paris, and Oxford), where Islamic works were taught in Latin translations. Among the Muslim scholars well-known in the centres of learning throughout the world were al-Rāzī (Rhazes), Ibn Sīnā (Avicenna), Ibn Rushd (Averroes), al-Khwārizmī and Ibn Khaldūn. Muslim scholars such as these and others produced original works in many fields. Many of them possessed encyclopaedic knowledge and distinguished themselves in many disparate fields of knowledge.

The Center for Muslim Contribution to Civilization was established in order to acquaint non-Muslims with the contributions Islam has made to human civilization as a whole. The Great Books of Islamic Civilization Project attempts to cover the first 800 years of Islam, or what may be called Islam's Classical Period. This project aims at making available in English and other European languages a wide selection of works representative of Islamic civilization in all its diversity. It is made up of translations of original Arabic works that were produced in the formative centuries of Islam, and is meant to serve the needs of a potentially large readership. Not only the specialist and scholar, but the non-specialist with an interest in Islam and its cultural heritage will be able to benefit from the series. Together, the works should serve as a rich source for the study of the early periods of Islamic thought.

In selecting the books for the series, the Center took into account all major areas of Islamic intellectual pursuit that could be represented. Thus the series includes works not only on better-known subjects such as law, theology, jurisprudence, history and politics, but also on subjects such as literature, medicine, astronomy, optics and geography. The specific criteria used to select individual books were these: that a book should give a faithful and comprehensive account of its field; and that it should be an authoritative source. The reader thus has at his disposal virtually a whole library of informative and enlightening works.

Each book in the series has been translated by a qualified scholar and reviewed by another expert. While the style of one translation will naturally differ from another as do the styles of the authors, the translators have endeavoured, to the extent it was possible, to make the works accessible to the common reader. As a rule, the use of footnotes has been kept to a minimum, though a more extensive use of them was necessitated in some cases.

This series is presented in the hope that it will contribute to a greater understanding in the West of the cultural and intellectual heritage of Islam and will therefore provide an important means towards greater understanding of today's world.

May God Help Us!

Muhammad bin Hamad Al-Thani
Chairman of the Board of Trustees

About this Series

This series of Arabic works, made available in English translation, represents an outstanding selection of important Islamic studies in a variety of fields of knowledge. The works selected for inclusion in this series meet specific criteria. They are recognized by Muslim scholars as being early and important in their fields, as works whose importance is broadly recognized by international scholars, and as having had a genuinely significant impact on the development of human culture.

Readers will therefore see that this series includes a variety of works in the purely Islamic sciences, such as Qurʾān, *ḥadīth*, theology, prophetic traditions (*sunna*), and jurisprudence (*fiqh*). Also represented will be books by Muslim scientists on medicine, astronomy, geography, physics, chemistry, horticulture, and other fields.

The work of translating these texts has been entrusted to a group of professors in the Islamic and Western worlds who are recognized authorities in their fields. It has been deemed appropriate, in order to ensure accuracy and fluency, that two persons, one with Arabic as his mother tongue and another with English as his mother tongue, should participate together in the translation and revision of each text.

This series is distinguished from other similar intercultural projects by its distinctive objectives and methodology. These works will fill a genuine gap in the library of human thought. They will prove extremely useful to all those with an interest in Islamic culture, its interaction with Western thought, and its impact on culture throughout the world. They will, it is hoped, fulfil an important rôle in enhancing world understanding at a time when there is such evident and urgent need for the development of peaceful coexistence.

This series is published by the Center for Muslim Contribution to Civilization, which serves as a research centre under the patronage of H.H. Sheikh Muhammad bin Hamad al-Thani, the former Minister of Education of Qatar who also chairs the Board of Trustees. The Board is comprised of a group of prominent scholars. These include His Eminence Sheikh Al-Azhar, Arab Republic of Egypt, and Dr Yousef al-Qaradhawi, Director of the Sira and Sunna Research Center. At its inception the Center was directed by the late Dr Muhammad Ibrahim Kazim, former Rector of Qatar University, who established its initial objectives.

The Center was until recently directed by Dr Kamal Naji, the Foreign Cultural Relations Advisor of the Ministry of Education of Qatar. He was assisted by a Board comprising a number of academicians of Qatar University, in addition to a consultative committee chaired by Dr Ezzeddin Ibrahim, former Rector of the University of the United Arab Emirates. A further committee acting on behalf of the Center has been the prominent university professors who act under the chairmanship of Dr Raji Rammuny, Professor of Arabic at the University of Michigan. This committee is charged with making known, in Europe and in America, the books selected for translation, and in selecting and enlisting properly qualified university professors, orientalists and students of Islamic studies to undertake the work of translation and revision, as well as overseeing the publication process.

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Translator's Introduction

When I accepted the challenge of translating Omar Al-Khayyam's book from Arabic to English, I told myself this would be an easy job to do. After all, I consider myself to be good at both mathematics and the Arabic language. The shock came when I discovered that first I had to translate from the Arabic of Omar Al-Khayyam to my Arabic. That was the difficult part of my job. Another shock was to discover my ignorance of the scientific achievements of great scientists such as Omar Al-Khayyam. Going through the book, reading theorems, proofs and remarks, I realized that I was reading the work of a great mathematician. The preciseness of the statements and the accuracy of the proofs made me think that I was reading an article in a high-ranking recent mathematical journal.

Omar Al-Khayyam was born in the middle of the eleventh century in the city of Nishapur. He was a poet and a mathematician. I read some of his poetry when I was an undergraduate student. At that time I did not know that he was a mathematician and had written books on mathematics.

Omar Al-Khayyam's book mainly deals with equations of degree at most three:

$$ax^3 + bx^2 + cx + d = 0$$

including all cases where some of the integer coefficients a, b, c, d equal zero.

At the time of Omar Al-Khayyam, the two equations

$$ax^3 + bx^2 + cx + d = 0$$
$$ax^3 + bx^2 + cx = d$$

were regarded as two different cases of equations of degree three.

Al-Khayyam's book studies and presents the following:

1. Third degree equations that can be reduced to equations of degree two.
2. Third degree equations that consist of three terms.
3. Third degree equations that consist of four terms.
4. Equations that involve the reciprocal of the unknown (variable).
5. The problem of dividing a quarter of a circle into two parts with a given ratio.
6. A discussion of some results of Abu-Aljood Ben Al-Laith.

Omar Al-Khayyam used geometry, especially conic sections, to prove his results.

I learned so much from the project of translating Omar Al-Khayyam's book – *Algebra wa Al-Muqabala* (Algebra and equations). Just after I finished translating the book, I met Professor Roshdi Rashid, in Amman at a conference on the history

of Arab sciences. He drew my attention to the fact that there are actually two different manuscripts of Al-Khayyam's book. One copy is in Aleppo, the one that I translated into English. The second copy is in France, the one he translated into English. I thank Professor Rashid very much for his comments and the valuable information he supplied me with.

I wish to thank Professor Raji Rammuny who suggested my name for the project of translating Al-Khayyam's book, thus offering me the chance to explore part of the work of a great scholar of my culture.

I thank all those who helped me in my task in one way or another, in particular my students S. Awamlah and A. Khawalda, who helped me in drawing the graphs for the book.

Algebra wa Al-Muqabala

An Essay by the Uniquely Wise 'Abel Fath Omar Bin Al-Khayyam on Algebra and Equations

One of the educational notions needed in the branch of philosophy known as mathematics is the art of algebra and equations, invented to determine unknown numbers and areas. It involves problems that reflect difficult propositions; most people studying such problems have been unable to solve them. In the case of the ancient ones [researchers], none of their work has reached us, either because they did not solve these problems despite trying, or because they did not need to solve them, or simply because their work was lost.

As for the modern ones, Mahani tried to analyze algebraically the proposition used by Archimedes as a postulate in proposition four of the second article of his algebra book on spheres and cylinders. In his analysis, he discovered equations involving squares and cubes of numbers that he could not solve, though he thought deeply about them. So, he concluded that such equations are impossible to solve. No one could solve such equations, until the genius "Abu Jafar Al-Khazin" solved them using conic sections.

Later on, a group of geometers needed certain classes of these equations. Some of them (the geometers) solved certain types of these problems. But none of them did any sound work concerning the classification and sub-classification of such problems or the proof of any, except for two classes that I will mention later in this book. And I was, and remain, very keen to classify the problems and indicate (by proof) those that can be solved and those that are impossible to solve, since I knew the very need of it in solving other (open) problems.

I was unable to devote myself completely to achieving this worthy task, or to pursuing my ideas generally, because the demands of my daily life were a necessary diversion.

We have been afflicted in our time by a lack of scientists, except for a particular group, few in number but many in the troubles that beset them, whose concern is to exploit any gleam of trouble-free time to achieve and articulate some branches of science.

Many of those who pretend to be wise men in our time defraud the truth with falsity and do not seek to move forward the frontiers of knowledge, preferring instead to use the little they know of science for low materialistic goals. And once they meet someone who sincerely wishes to acquire facts and who prefers truth, trying to reject falsity and avoid deception, they make fun of him and ridicule him. So may God be our helper and our comforter.

God gave me the opportunity of being with our unique great master, the head judge, the scholar Imam Abi Tahir, may God keep his high position, and keep silent his enviers and enemies. Once I despaired to meet someone like him, perfect in every virtue, theoretical or practical, who can work deeply in science, verify others' work and seeks the welfare of everyone of his kind; I was so delighted to see him. I achieved fame through his companionship. My affairs were glorified by his illuminations, and my support intensified through his grace. It was my opportunity to benefit from my new status.

So I started to summarize what I can verify of deep knowledge so as to be closer to the master (Abi Tahir). Since the priority is mathematics, I started to list the algebraic propositions.

I adhered to the guidance of God, and I entreated God to grant me success in verifying my scientific research and the important research of those before me, grasping the most trustworthy of God's protection. For He is the one who answers our prayers and on whom we depend.

With the help of God, and with his gentle guidance, I say:

The study of algebra and equations is a scientific art. The ingredients are the absolute numbers and unknown measurable quantities, which are related to a known quantity. Each known thing is either a quantity or a unique relation that can be determined by careful examination.

By quantities we mean continuous quantities, and they are of four types: line, surface, solid, and time – as mentioned briefly in *Categories*, the book of Aristotle, and in detail in his other book, *Metaphysics*. Some (researchers) consider place to be a continuous quantity of the same type as surface. This is not the case, as one can verify. The truth is: space is a surface with conditions, whose verification is not part of our goal in this book. It is not usual to mention time as an object in algebraic problems. But if it were mentioned, it would be quite acceptable.

It is a practice of the algebraists (in their work) to call the unknown to be determined an object (variable), and the product of the object by itself a square (maal). The product of the object by its square is called a cube, and the product of the square by the square: square–square (maal–maal); the product of cube by square: square–cube, and the product of cube by cube: cube–cube, and so on.

It is known from the *Elements*, the book of Euclid, that these ranked products are all proportional in the sense that the ratio of one to the root is as the ratio of root to square, is as the ratio of square to cube. So the ratio of the numbers to roots is as the ratio of roots to squares, is as the ratio of squares to cubes, is as the ratio of cubes to square–squares, and so on.

It has to be clear that for anyone to be able to understand this essay, he has to be acquainted with the two books of Euclid (the *Elements* and the *Data*) and two chapters of the book of Apollonius on *Cones*. Whoever lacks knowledge of any of these three references, will not understand this essay. I have taken pains in trying not to refer to any article or book except those three books.

Algebraic solutions are achieved using equations. I mean, as is well known, by equating the ranks (powers) one with the other. If an algebraist uses square–square in areas, then this would be figuration not fact, since it is impossible for square–square to exist in measurable quantities. What we get in measurable quantities is first one dimension, which is the "root" or the "side" in relation to its square; then two dimensions, which represent the surface and the (algebraic) square representing the squared surface (rectangle); and finally three dimensions, which represent the solid. The cube in quantities is the solid bounded by six squares (parallelepiped), and since there is no other dimension, the square of the square does not fall under measurable quantities let alone higher powers.

If it is said that the square of the square is among measurable quantities, this is said with reference to its reciprocal value in problems of measurement and not because it in itself is measurable. There is a difference between the two cases.

The square of the square is, therefore, neither essentially nor accidentally a measurable quantity and is not as even and odd numbers, which are accidentally included in measurable quantities, depending on the way in which they represent continuous measurable quantities as discontinuous.

Of the four (geometrical) equations that involve the absolute numbers, sides, squares and cubes, the books of the algebraists contain only three of these equations involving numbers, sides and squares. But we will give the methods by which one can determine the unknown using equations that involve the four measurable quantities that we mentioned, I mean: the number, the object, the square and the cube. Whatever can be proved using the properties of the circle, I mean using the two books of Euclid, the *Elements* and the *Data*, will be given simpler proofs. But those that cannot be proved except by using conic sections will be proved using two articles on conics.

As for the proofs of these types (of problems): if the problem is concerned only with the absolute number, then we cannot (in general) supply the proofs (and no one who works in this industry can). Hopefully, someone who comes after us will be able to (supply the proofs). As for the first three ranks: number, object and square, we will supply the proofs, and I may refer to numerical proofs of problems that can be proved using the book of Euclid (the *Elements*).

You have to know that the geometrical proofs of these problems do not dispense the numerical proofs if the topic is the number not the measurable quantities. Can you not see that Euclid proved certain equations to find the required rational measurable quantities in chapter five of his book (the *Elements*), and then resumed his proof of such problems, in chapter seven of his book, to determine these required ratios, if the topic is some number.

Equations involving these four types are either simple or multi-term equations. The simple equations are of six types:

(1) Number equals root
(2) Number equals square
(3) Number equals cube
(4) Roots equal square
(5) Squares equal cube
(6) Roots equal cube.

Three of these six equations are mentioned in the books of the algebraists. They (the algebraists) said: the ratio of the object to the square is as the ratio of the square to the cube. So equating the square to the cube is as equating the object to the square. Further, the ratio of the number to the square is as the ratio of the root to the cube. So it follows that the equation of the number and the square is as the equation of the root and the cube. They (the algebraists) did not prove that using geometry.

As for the number that equals the (volume) cube, there is no way to determine it (the number) except through mathematical induction. If a geometrical method is to be used to determine the number, then it can only be done through conic sections.

As for multi-term equations, they are of two classes: three-term equations and four-term equations. The three-term equations are of twelve types. The first three of them are:

(1) Square and a root equal a number
(2) Square and a number equal root
(3) Root and a number equal square.

These three have been mentioned in the algebraists' books, including their proofs using geometry, not using numbers. The second three (of the three-term equations) are:

(1) Cube and square equal root
(2) Cube and root equal square
(3) Cube equals root and square.

The algebraists said that these three equations are equivalent to the first three, each to the corresponding one. I mean: cube and root equal square is equivalent to square and number equal root. And the other two are the same. They (the algebraists) did not prove them if the topic (unknown) of the problem is area. But they did solve them if the unknown is number, as is clear in the book of *Elements*. I will prove the geometrical ones.

The other six types of the twelve types are:

(1) Cube and root equal number
(2) Cube and number equal root
(3) Number and root equal cube
(4) Cube and square equal number
(5) Cube and number equal square
(6) Number and square equal cube.

These six types were never mentioned in their (the algebraists') books, except for one, where the proof was not complete. I will prove all of these types using geometry, not numeric. The proofs of these six types can only be deduced through the properties of conic sections.

As for the four-term equations, they consist of two groups. The first group, where three terms equal one term, contains four types:

(1) Cube, square and root equal number
(2) Cube, square and number equal root
(3) Cube, root and number equal square
(4) Cube equals root, square and number.

The second group, in which two terms equal two terms, is of three types:

(1) Cube and square equal root and number
(2) Cube and root equal square and number
(3) Cube and number equal root and square.

These are the seven types of the four-term equations, none of which can be solved except through geometry.

One of those (algebraists) who lived before us needed one type of one part of these equations, as I will mention. The proof of these types cannot be produced except through conic sections. We will prove the twenty-five types of these equations, one by one, asking help from God, for those who sincerely depend on Him will get help and guidance.

First type of simple equations: Root equals a number ($ax = b$)

The root is necessarily known; this applies to numbers and areas.

Second type: Number equals a square ($x^2 = b$)

So the square is known, being equal to the known number. There is no way to find the root except by trial. Those who know that the root of twenty-five is five know that by induction, not through a deduced formula, and one need pay no attention to people who differ in this matter. People of India have ways of finding the side of a square (knowing the area) and of a cube (knowing the volume) and these methods are based on simple induction, that depends on knowing the squares of the nine (numbers). I mean the square of one, two, three, and their products (that is to say the product of two by three and so on). We have written an article in which we prove the validity of these methods and show how it leads to the required results. We enriched its types, I mean by finding the sides of square-square, square-cube, and cube-cube, and so on. We were the first to do that. These proofs are numerical ones and are based on the numerical part of the book of *Elements*.

The following is the proof of the second type using geometry:

Draw the line <u>ab</u> whose length equals the given number and a line <u>ac</u> (of unit length) perpendicular to <u>ab</u> and then complete the surface (rectangle) <u>ad</u>. The area of the surface <u>ad</u> is equal to the given number. We construct a square whose area equals the area of <u>ad</u>; call it <u>h</u>, as shown by Euclid, proposition <u>zd</u>, article <u>b</u>